Integration

WSA

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$$

2) (b) $\int (12x^3 - 4x)\,dx = \frac{12x^4}{4} - \frac{4x^2}{2} + c = 3x^4 - 2x^2 + c$

(f) $\int x(x^2 - 3)\,dx = \int x^3 - 3x\,dx = \frac{1}{4}x^4 - \frac{3}{2}x^2 + c$

(j) $\int \left(x - \frac{1}{x^3}\right)dx = \int (x - x^{-3})\,dx = \frac{1}{2}x^2 + \frac{1}{2}x^{-2} + c$

(4) (b) $\int (2r + \sqrt{r})\,dr = \int (2r + r^{\frac{1}{2}})\,dr = \frac{2r^2}{2} + \frac{2r}{3}^{3/2} + c$

(f) $\int \left(\frac{1}{2}x^2 - x^{\frac{3}{2}}\right)dx = \frac{1}{6}x^3 - \frac{2x}{5}^{5/2} + c$

(j) $\int (4 - y^{\frac{7}{4}})\,dy = 4y - \frac{4}{7}y^{\frac{11}{4}} + c$

(5) (b) $y = x^6 - x^3 + 2x - 5$

$\int y\,dx = \int (x^6 - x^3 + 2x - 5)\,dx = \frac{1}{7}x^7 - \frac{1}{4}x^4 + x^2 - 5x + c$

(e) $y = (x^2 - 4)(2x + 3) = 2x^3 + 3x^2 - 8x - 12$

$\int y\,dx = \int (2x^3 + 3x^2 - 8x - 12)\,dx = \frac{1}{2}x^4 + x^3 - 4x^2 - 12x + c$

(6) (e) $\frac{dy}{dx} = 2x - \frac{3}{\sqrt{x}}$, $\frac{dy}{dx} = 2x - 3x^{-\frac{1}{2}}$

$y = \int (2x - 3x^{-\frac{1}{2}})\,dx = x^2 - 6x^{\frac{1}{2}} + c$

(h) $\frac{dy}{dx} = \frac{2}{x^3}(5 - x) = 10x^{-3} - 2x^{-2}$

$y = \int (10x^{-3} - 2x^{-2})\,dx = -5x^{-2} + 2x^{-1} + c$

WSB

(d) $\frac{dy}{dx} = 7 - 5x - x^3$ $\quad y=0 \quad x=2$

$y = \int (7 - 5x - x^3)\,dx = 7x - \frac{5}{2}x^2 - \frac{1}{4}x^4 + c$

$y=0,\ x=2 \Rightarrow 0 = 7(2) - \frac{5}{2}(2)^2 - \frac{1}{4}(2)^4 + c$

$0 = 14 - 10 - 4 + c \Rightarrow \underline{c=0}$

$y = 7x - \frac{5}{2}x^2 - \frac{1}{4}x^4$

(f) $\frac{dy}{dx} = 3 - \sqrt{x} = 3 - x^{\frac{1}{2}}$

$y=8$ at $x=4$

$y = \int 3 - x^{\frac{1}{2}}\,dx = 3x - \frac{2}{3}x^{\frac{3}{2}} + c$

$y=8$ when $x=4 \Rightarrow 8 = 3(4) - \frac{2}{3}(4)^{\frac{3}{2}} + c$

$8 = 12 - \frac{2}{3}(8) + c$

$-4 + \frac{16}{3} = c \Rightarrow \underline{c = \frac{4}{3}}$

$y = 3x - \frac{2}{3}x^{\frac{3}{2}} + \frac{4}{3}$

(Q5) $(-1,4)$ point. $f'(x) = 2x^3 - x - 8, \Rightarrow f(x) = \int f'(x)\,dx$

$f(x) = \int (2x^3 - x - 8)\,dx = \frac{1}{2}x^4 - \frac{1}{2}x^2 - 8x + c$

use $(-1,4)$ $\Rightarrow 4 = \frac{1}{2}(-1)^4 - \frac{1}{2}(-1)^2 - 8(-1) + c$

$4 = \cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} + 8 + c$

$\Rightarrow \underline{c = -4}$

$f(x) = \frac{1}{2}x^4 - \frac{1}{2}x^2 - 8x - 4$

In General if $n$ is an integer

$(-1)^{2n} = 1$

$(-1)^{2n+1} = -1$

$\underline{x=2} \Rightarrow y = f(2) = \frac{1}{2}(2)^4 - \frac{1}{2}(2)^2 - 8(2) - 4$

$= 8 - 2 - 16 - 4 = -14$

$m = f'(2) = 2(2)^3 - 2 - 8$

$= 16 - 2 - 8 = \underline{\underline{6}}$

$(2,-14) \quad m=6$

$y - y_1 = m(x - x_1)$

$\underline{y + 14 = 6(x-2)}$

(8) $\frac{dy}{dx} = 3x^2 + kx \Rightarrow$ $y = \int 3x^2 + kx \, dx$

$(1,6)$ $(2,1)$

$y = x^3 + \frac{k}{2}x^2 + c$

$(1,6) \Rightarrow 6 = 1^3 + \frac{k}{2}(1)^2 + c \Rightarrow 6 = 1 + \frac{k}{2} + c \Rightarrow 10 = k + 2c$

$(2,1) \Rightarrow 1 = 2^3 + \frac{k}{2}(2)^2 + c \Rightarrow 1 = 8 + 2k + c \Rightarrow -7 = 2k + c$

$10 = k + 2c$ —— ①

$-7 = 2k + c$ $(\times 2)$

$-14 = 4k + 2c$ —— ②

$-21 = 3k \Rightarrow \underline{k = -7}$ sub into ① $\Rightarrow$ $10 = -7 + 2c$

$2c = 17 \Rightarrow \underline{c = \frac{17}{2}}$

$y = x^3 + \frac{k}{2}x^2 + c \Rightarrow \underline{y = x^3 - \frac{7}{2}x^2 + \frac{17}{2}}$

## WSC

(Q3) $y = f(x)$ $(3, 22)$

$f'(x) = 3x^2 + 2x - 5$

(a) $f(x) = \int 3x^2 + 2x - 5 \, dx = x^3 + x^2 - 5x + c$

$(3,22) \Rightarrow$ $22 = (3)^3 + 3^2 - 5(3) + c$

$22 = 27 + 9 - 15 + c \Rightarrow \underline{c = 1}$

$\underline{f(x) = x^3 + x^2 - 5x + 1}$

(b) $g(x) = (x+3)(x-1)^2 = (x+3)(x^2 - 2x + 1) = x^3 - 2x^2 + x + 3x^2 - 6x + 3$

$g(x) = x^3 + x^2 - 5x + 3$

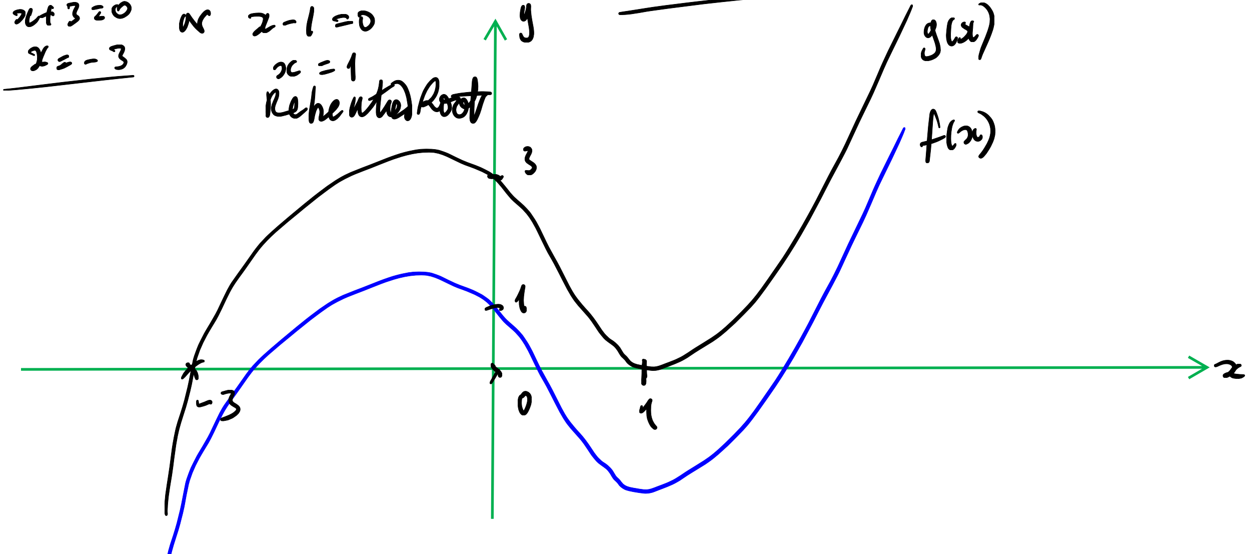
Crosses $x$-axis when $y = 0$ $(x+3)(x-1)^2 = 0$

$x + 3 = 0$ or $x - 1 = 0$

$\underline{x = -3}$ $x = 1$

Repeated Root

Therefore $\underline{g(x) = f(x) + 2}$



(8) $y = f(x)$ $\qquad$ $P(-2, 0)$

$f'(x) = 3x^2 - 2x - 3$

(a) $f(x) = \int (3x^2 - 2x - 3)\,dx = x^3 - x^2 - 3x + c$

$P(-2,0)$ $\qquad 0 = (-2)^3 - (-2)^2 - 3(-2) + c$

$0 = -8 - 4 + 6 + c \Rightarrow \underline{c = 6}$

$f(x) = x^3 - x^2 - 3x + 6$

(b) Eqn of tangent $\underline{x=1}$ $\qquad m = f'(1) = 3(1)^2 - 2(1) - 3 = 3 - 2 - 3 = \underline{\underline{-2}}$

$m = -2$ $\qquad$ when $x = 1$ $\qquad y = f(1) = 1^3 - 1^2 - 3(1) + 6$

$y = \cancel{1} - \cancel{1} - 3 + 6 = \underline{\underline{3}}$ $\quad (1, 3)$

$m = -2$ $\qquad (1, 3)$

$y - y_1 = m(x - x_1)$

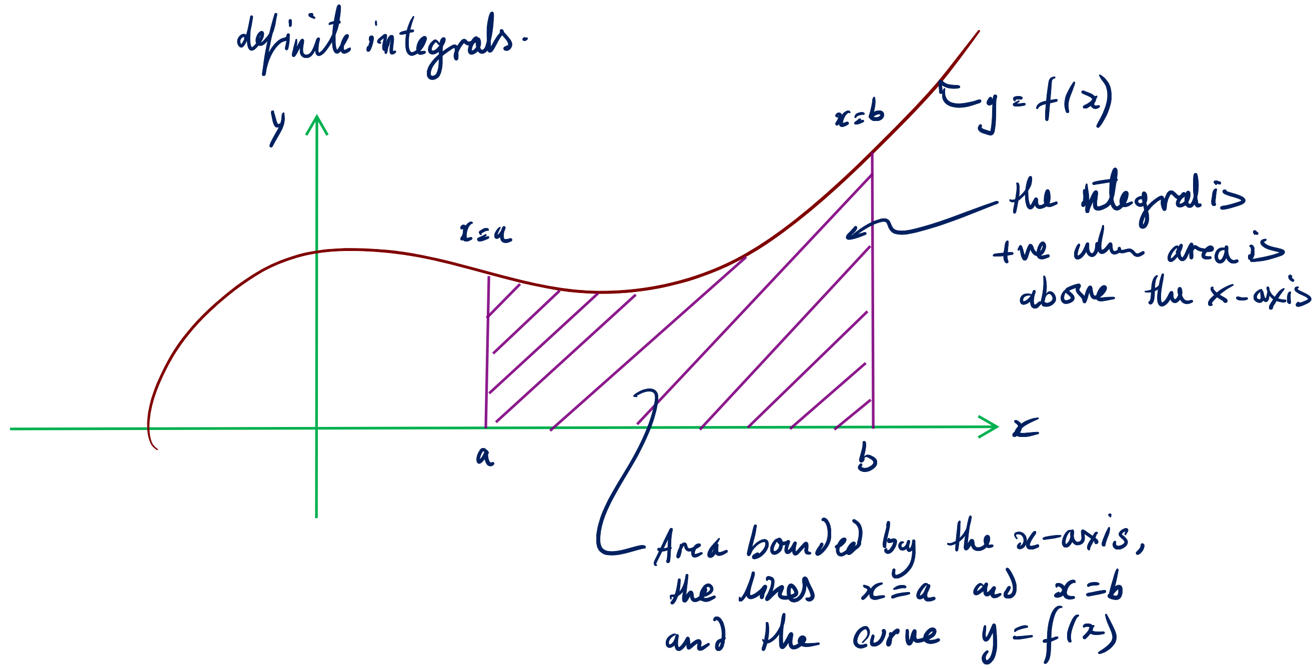
$y - 3 = -2(x - 1)$

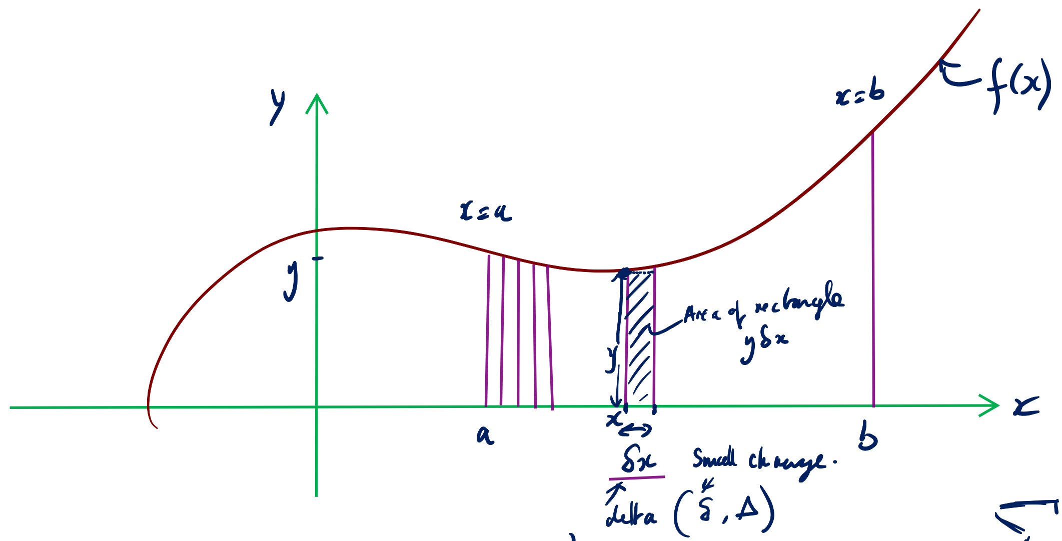
$y - 3 = -2x + 2$

$y = -2x + 5$

As required.

definite integrals.






$\frac{d}{\delta}$, $\Sigma$ sum of

$\sum$ sigma mean sum of.

$$\text{Area} = \lim_{\delta x \to 0} \sum_{x=a}^{b} y\,\delta x$$

$$\text{Area} = \int_a^b y\,dx$$

## C2 integration WS A

1(e) $\int_1^2 (x^2 - 8x - 3)\,dx = \frac{1}{3}x^3 - 4x^2 - 3x\Big]_1^2$

$= \frac{1}{3}(2)^3 - 4(2)^2 - 3(2) - \left[\frac{1}{3}(1)^3 - 4(1)^2 - 3(1)\right]$

$= \frac{8}{3} - 16 - 6 - \frac{1}{3} + 4 + 3$

$= \frac{7}{3} - 15 = -\frac{38}{3} = -12\frac{2}{3}$

(h) $\int_{-2}^{-1} (5 + x^2 - 4x^3)\,dx = 5x + \frac{1}{3}x^3 - x^4\Big|_{-2}^{-1}$

$= 5\big((-1) - (-2)\big) + \frac{1}{3}\big((-1)^3 - (-2)^3\big) - \big((-1)^4 - (-2)^4\big)$

$= 5(1) + \frac{1}{3}(-1 + 8) - (1 - 16)$

$= 5 + \frac{7}{3} + 15 = 20 + \frac{7}{3} = 22\frac{1}{3}$

Q4) (f) $\int_2^3 \frac{1 - 6x^3}{3x^2}\,dx = \int_2^3 \frac{1}{3x^2} - \frac{6x^3}{3x^2}\,dx = \int_2^3 \frac{1}{3}x^{-2} - 2x\,dx$

$= -\frac{1}{3}x^{-1} - x^2\Big]_2^3 = -\frac{1}{3}\left(\frac{1}{3} - \frac{1}{2}\right) - \left(3^2 - 2^2\right)$

$= -\frac{1}{3}\left(\frac{2-3}{6}\right) - (9 - 4)$

$= \frac{1}{18} - 5 = -4\frac{17}{18}$

**Q7** (a) $y = 4 - x^2$ crosses $x$-axis when $y=0$

$4 - x^2 = 0$

$(2-x)(2+x) = 0$

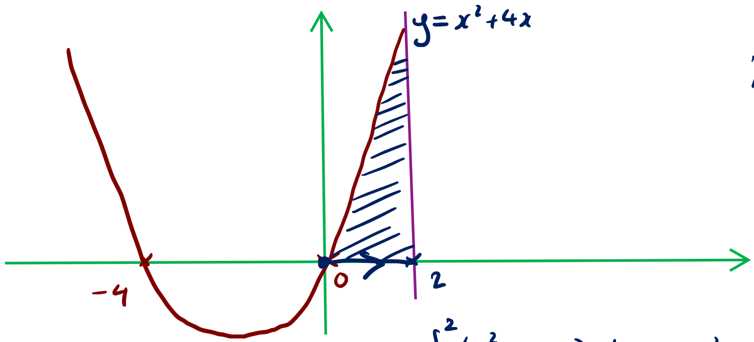
$x = 2$ or $x = -2$

$(2,0)$ $(-2,0)$

(b) $\text{Area} = \int_{-2}^{2} (4 - x^2)\,dx = 4x - \frac{1}{3}x^3 \Big|_{-2}^{2} = 4(2-(-2)) - \frac{1}{3}(2^3 - (-2)^3)$

$= 16 - \frac{1}{3}(8+8) = 16 - \frac{16}{3} = \frac{32}{3} = 10\frac{2}{3}$ units$^2$

(9)




$y = x^2 + 4x$

crosses $x$-axis when

$y=0$ $x^2 + 4x = 0$

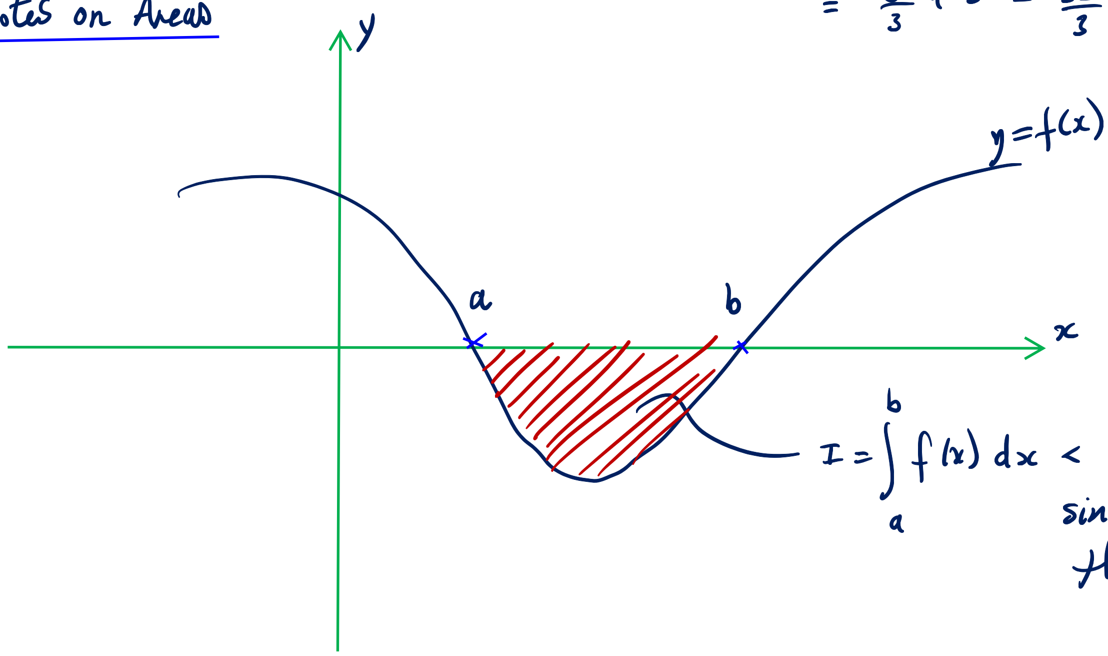
$x(x+4) = 0$

$x = 0$ $x = -4$

$\int_0^2 (x^2 + 4x)\,dx = \frac{1}{3}x^3 + 2x^2 \Big|_0^2 = \frac{1}{3}(2)^3 + 2(2)^2 - 0$

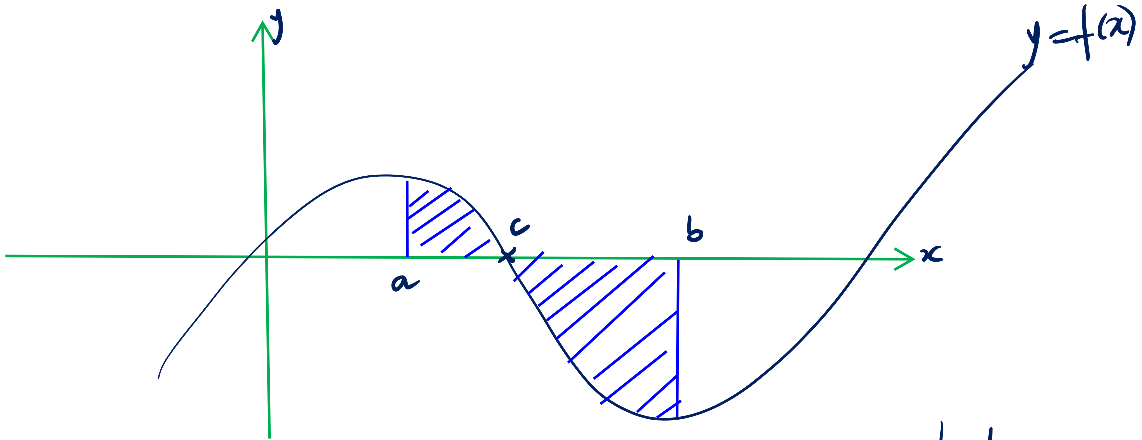
$= \frac{8}{3} + 8 = \frac{32}{3} = 10\frac{2}{3}$ units$^2$

**Notes on Areas**

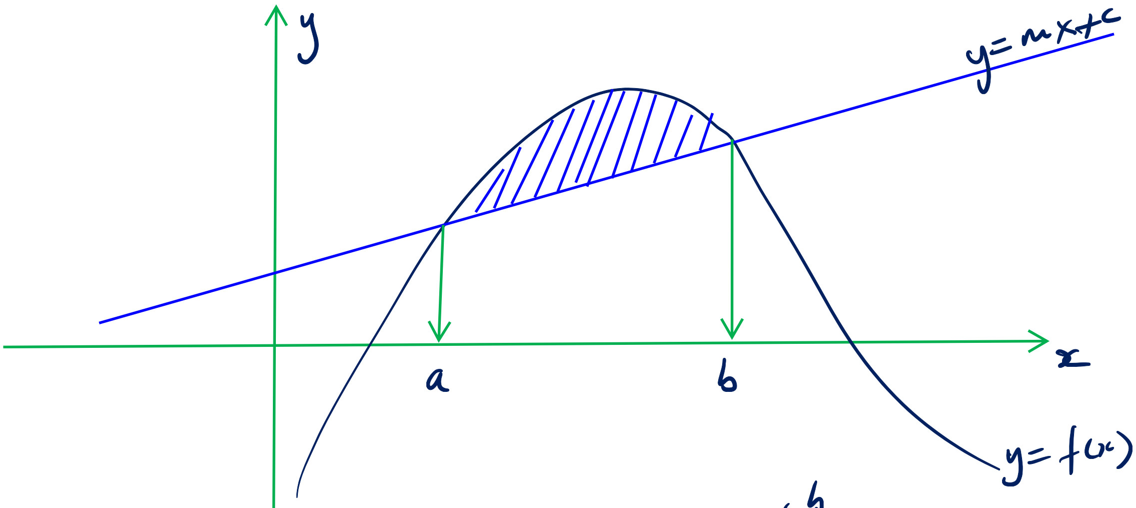



$I = \int_a^b f(x)\,dx < 0$ Negative.

since Area lies below the $x$-axis

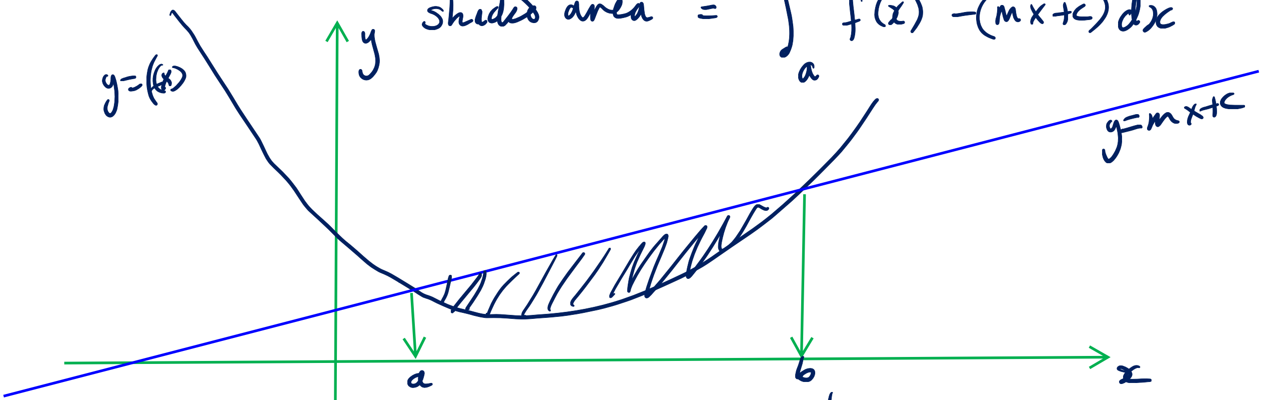



shaded Area $= \int_a^c f(x)\,dx + \left| \int_c^b f(x)\,dx \right|$

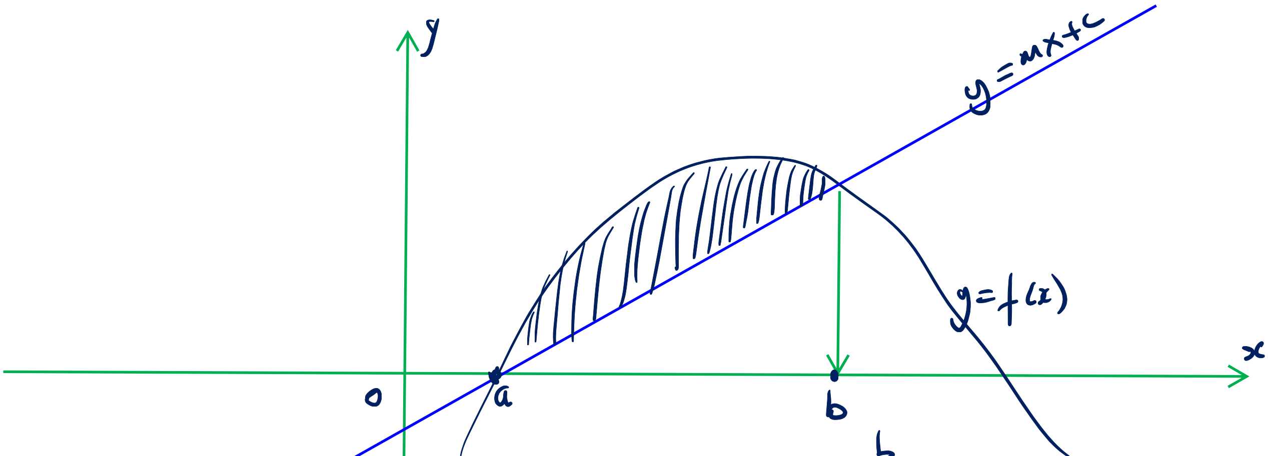



shaded area $= \int_a^b f(x)\,dx$ - Area of trapezium

shaded area $= \int_a^b f(x) - (mx+c)\,dx$

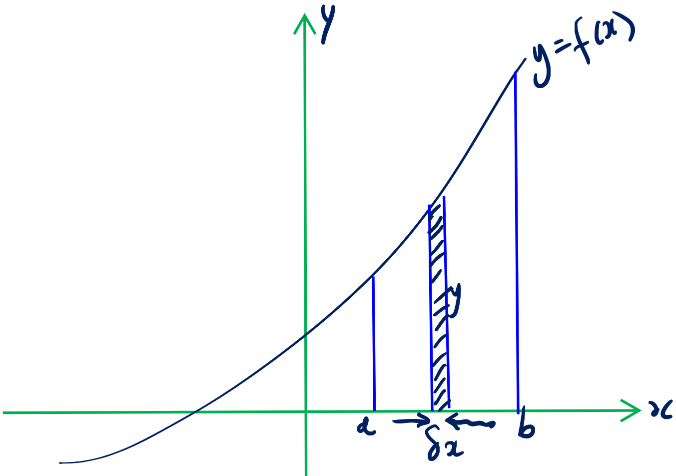



shaded area $= \int_a^b \underbrace{(mx+c)}_{\text{line}} - \underbrace{f(x)}_{\text{curve}}\,dx$



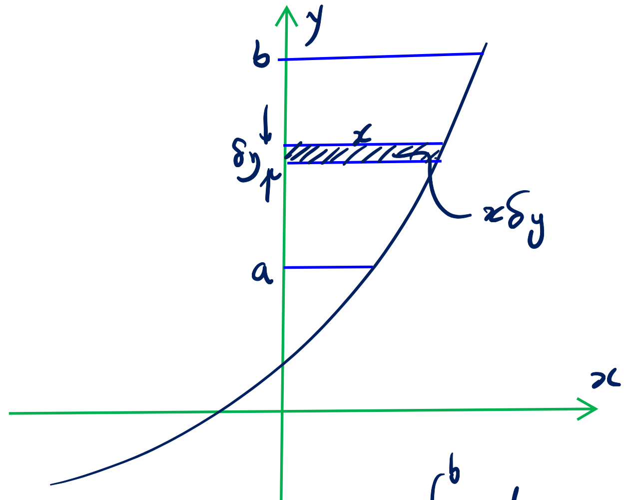

shaded area $= \int_a^b f(x) - (mx+c)\,dx$

shaded area $= \int_a^b f(x)\,dx$ - Area of $\Delta$.




$$\text{Area} = \int_a^b f(x)\,dx$$

Area bounded the x-axis, the curve $y=f(x)$, the lines $x=a$ and $x=b$.




$$\text{Area} = \int_a^b x\,dy$$

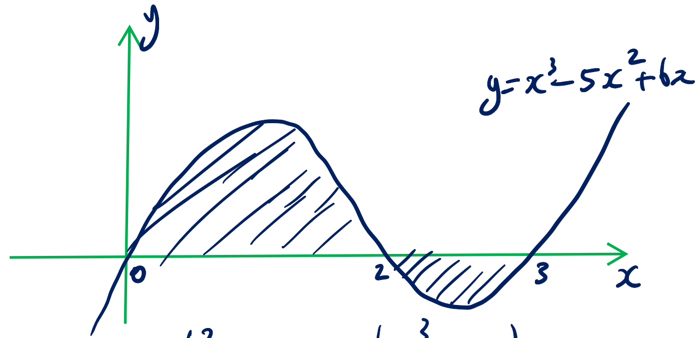
Area bounded the y-axis, the curve $x=f(y)$, the lines $y=a$ and $y=b$.

WSA continued.

Q11) $y = x^3 - 5x^2 + 6x$

(a) Crosses x-axis when $y=0$

$$\Rightarrow x^3 - 5x^2 + 6x = 0$$
$$x(x^2 - 5x + 6) = 0$$
$$x(x-3)(x-2) = 0$$
$$x = 0 \text{ or } x = 3 \text{ or } x = 2$$



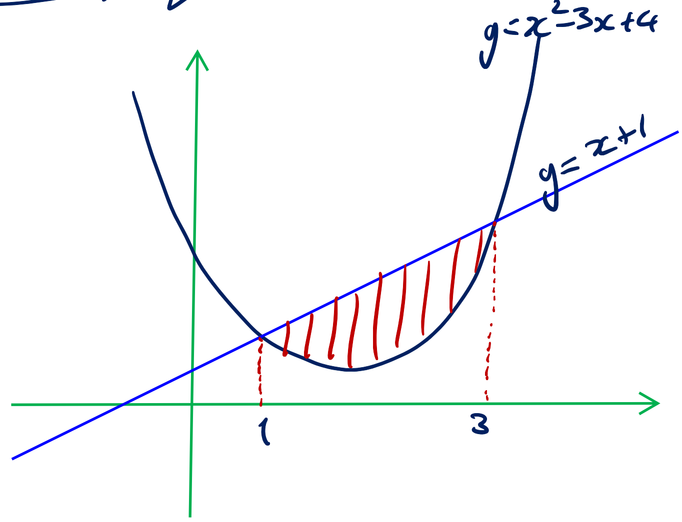

$$\text{Area} = \int_0^2 y\,dx + \left|\int_2^3 y\,dx\right|$$

$$\int_a^b x^3 - 5x^2 + 6x\,dx = \frac{1}{4}x^4 - \frac{5}{3}x^3 + 3x^2 \Big|_a^b$$

$$\text{Shaded Area} = \frac{1}{4}(2)^4 - \frac{5}{3}(2)^3 + 3(2)^2 - 0 + \left|\frac{1}{4}(3)^4 - \frac{5}{3}(3)^3 + 3(3)^2 - \frac{8}{3}\right|$$

$$\frac{8}{3} + \left|\frac{9}{4} - \frac{8}{3}\right| = \frac{8}{3} + \frac{5}{12} = \frac{37}{12} = 3\frac{1}{12}\text{ units}^2$$

(12) Solve simultaneously
$$x^2 - 3x + 4 = x + 1$$
$$x^2 - 4x + 3 = 0$$
$$(x-3)(x-1) = 0$$
$$x = 1 \text{ or } 3$$



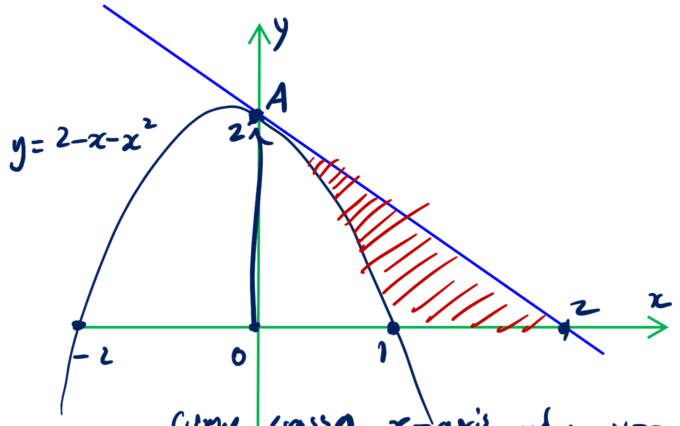

$$\text{Shaded Area} = \int_1^3 (\text{line} - \text{curve})\,dx$$
$$= \int_1^3 x + 1 - (x^2 - 3x + 4)\,dx$$
$$= \int_1^3 x + 1 - x^2 + 3x - 4\,dx$$

$$= \int_1^3 (-x^2 + 4x - 3)\,dx = -\tfrac{1}{3}x^3 + 2x^2 - 3x \Big|_1^3$$

$$= -\tfrac{1}{3}\left((3)^3 - (1)^3\right) + 2\left(3^2 - 1^2\right) - 3(3-1)$$

$$= -\tfrac{1}{3}(27-1) + 2(9-1) - 3(2)$$

$$= -\frac{26}{3} + 16 - 6 = \frac{4}{3} \text{ units}^2$$




$$y = x^n \quad \frac{dy}{dx} = n x^{n-1}$$

WSD

9 (a) Eqn of tangent

$$y = 2 - x - x^2 \Rightarrow \frac{dy}{dx} = -1 - 2x$$

at A when $x = 0$ $\quad \frac{dy}{dx} = -1 - 2(0) = -1 \Rightarrow m_t = -1$

find y at A $\Rightarrow y = 2 - 0 - 0^2 = 2 \quad (0,2)$

$$y = -x + 2$$

curve crosses x-axis when $y=0$

$2 - x - x^2 = 0$
$x^2 + x - 2 = 0 \Rightarrow (x-1)(x+2) = 0$
$x = 1,\ x = -2$

(b) $y = -x + 2$ crosses x-axis when $y = 0 \Rightarrow -x + 2 = 0 \Rightarrow x = 2$

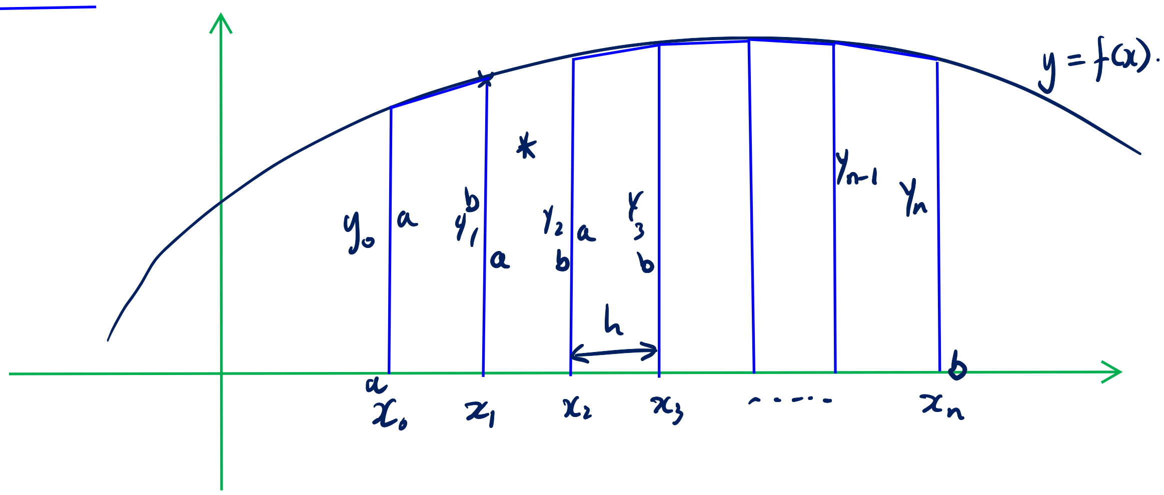
Area of $\Delta = \frac{1}{2} \times 2 \times 2 = 2$ units.

$$\text{Area under curve} = \int_0^1 y\,dx = \int_0^1 (2 - x - x^2)\,dx = 2x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 \Big|_0^1 = 2(1) - \tfrac{1}{2}(1)^2 - \tfrac{1}{3}(1)^3 - 0$$

$$= 2 - \frac{1}{2} - \frac{1}{3} = 2 - \frac{5}{6} = 1\frac{1}{6} \text{ units}^2$$

$$\text{Shaded Area} = 2 - 1\tfrac{1}{6} = \frac{5}{6} \text{ units}^2$$

## Trapezium Rule.



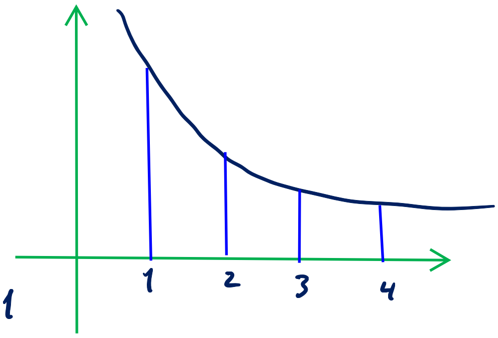

$$A = \tfrac{1}{2}(a+b)h$$

$$\text{Area under curve} = \tfrac{1}{2}(y_0 + y_1)h + \tfrac{1}{2}(y_1 + y_2)h + \tfrac{1}{2}(y_2 + y_3)h + \cdots \tfrac{1}{2}(y_{n-1} + y_n)h$$

$$\int_a^b f(x)\,dx \approx \tfrac{1}{2}\left[y_0 + y_n + 2(y_1 + y_2 + \cdots y_{n-1})\right]h.$$

WSC

(Q1) $y = \frac{3}{x}$



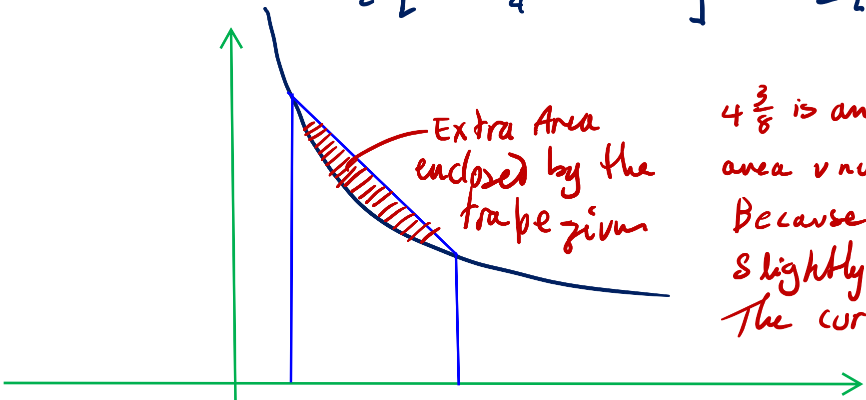
(a)

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $y$ | 3 | $\frac{3}{2}$ | 1 | $\frac{3}{4}$ |
| | $y_0$ | $y_1$ | $y_2$ | $y_3$ |

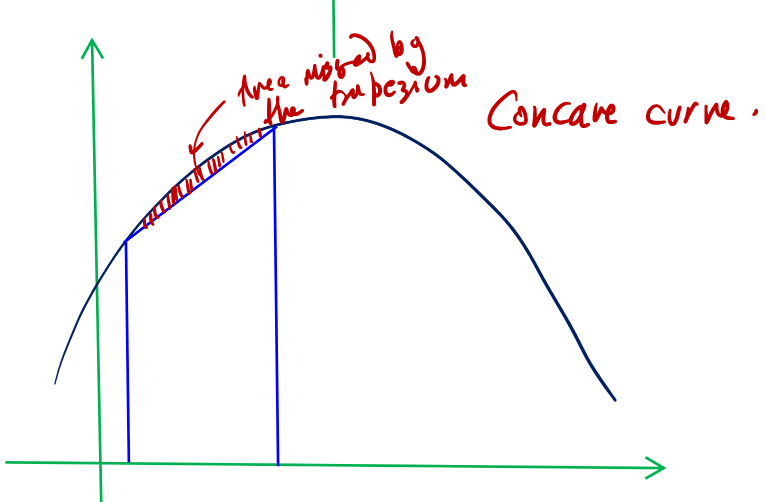
(b) Area under curve $= \frac{1}{2}\left[3 + \frac{3}{4} + 2\left(\frac{3}{2} + 1\right)\right]1$

$= \frac{1}{2}\left[3 + \frac{3}{4} + 3 + 2\right]1 = \frac{1}{2}\left[8\frac{3}{4}\right] = \underline{4\frac{3}{8} \text{ units.}}$

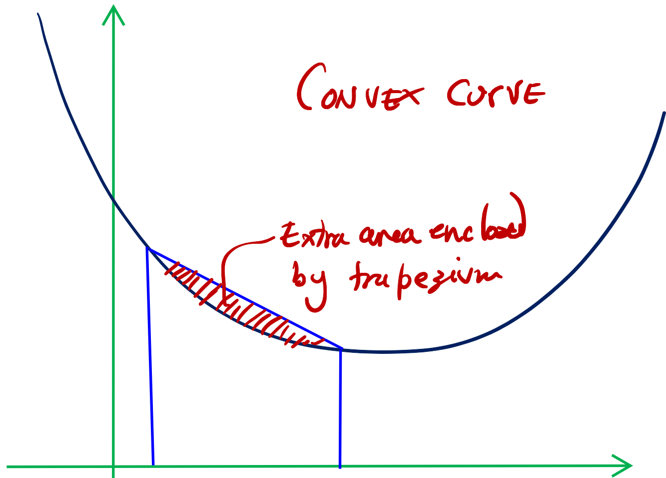
(c)



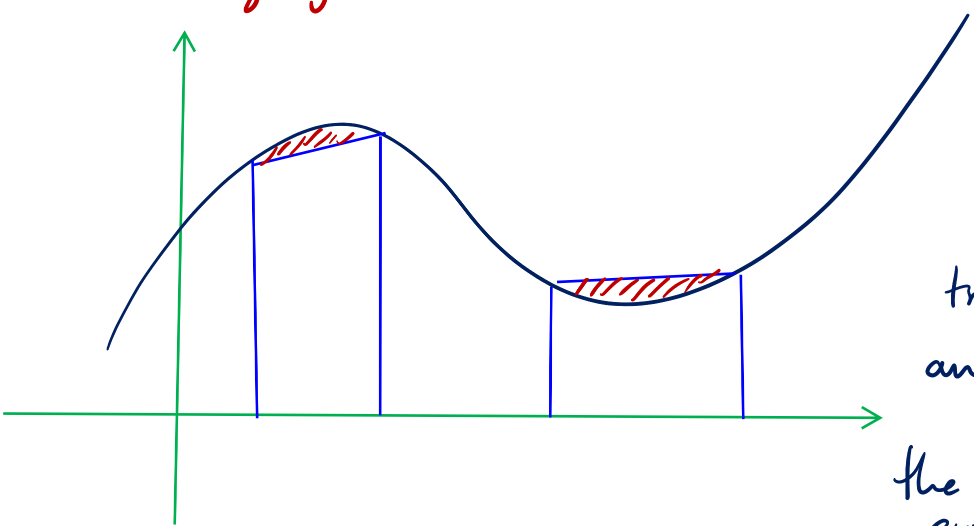

$4\frac{3}{8}$ is an overestimate to the area under the curve. Because each trapezium lies slightly above the curve. The curve is convex.




The trapezium rule gives an underestimate to the actual area under the curve. Since each trapezium lies slightly below the Concave curve



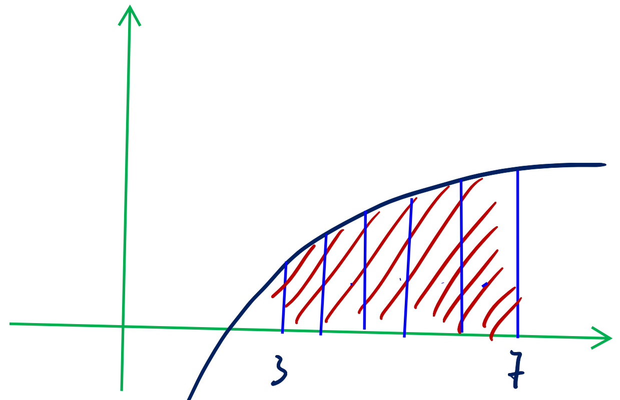

The trapezium rule gives an overestimate since each trapezium lies slightly above the convex curve.



In this case we can not say if the trapezium rule gives an over estimate or underestimate to the actual area under the curve.

(5)

$y = 2 - 3x^{-\frac{1}{2}}$




(a)

| $x$ | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|
| $y$ | 0.2679 | 0.5 | 0.6584 | 0.7753 | 0.8661 |

$\text{Area} = \frac{1}{2}\left[0.2679 + 0.8661 + 2(0.5 + 0.6584 + 0.7753)\right](1)$

{ we use 4 strips and not 5 as required }.

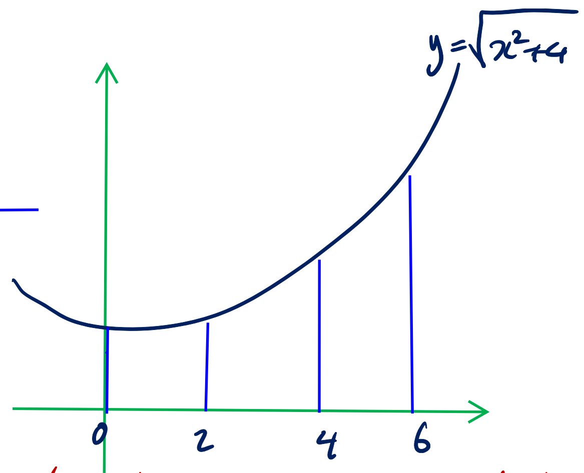
$\text{Area} = 2.5007$

(b) $\text{Area} = \int_3^7 2 - 3x^{-\frac{1}{2}}\,dx = \left. 2x - 6x^{\frac{1}{2}} \right]_3^7 = 2(7-3) - 6(\sqrt{7} - \sqrt{3})$

$= 8 - 6(\sqrt{7} - \sqrt{3}) = 2.5118$

WSD

Q2




(a)

| $x$ | 0 | 2 | 4 | 6 |
|---|---|---|---|---|
| $y$ | 2 | $2\sqrt{2}$ | $2\sqrt{5}$ | $2\sqrt{10}$ |

$\text{Area} = \frac{1}{2}\left[2 + 2\sqrt{10} + 2(2\sqrt{2} + 2\sqrt{5})\right]2$

$= 22.9 \text{ units}^2$

overestimate since each trapezium lies slightly above the curve.